Describing a Combinatorics Problem with a System of Polynomial Equations

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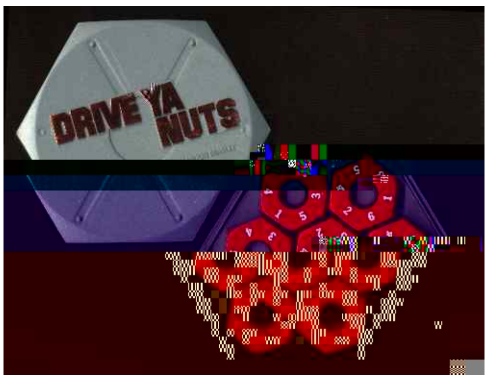
ABSTRACT
This paper provides a method to describe and solve a combinatorics problem using systems of polynomial equations. These systems, however, are too large to be solved by hand. The goal of this paper is to give the reader two techniques to solve these systems. The rst technique uses Buchberger's Algorithm to nd a Grobner basis for the system. The second technique addresses and solves the problem if nding a Grobner basis is computationally di cult.



1. INTRODUCTION
In 1970, Milton Bradley(c) created a game played on a hexagon-shaped grid called `Drive Ya Nuts.' The game consists of seven hexagonal nuts, each having a unique arrangement of the numbers one through six on each side. The object of the game is to arrange the nuts on the grid in such a way that adjacent sides of the nuts have matching numbers. Up to rotation of the entire game board, there are possible 235; 146; 240 ways to place the nuts. Suppose that we were able to go through every combination and check if it were a solution every second, then it would take 7.46 years to nd all solutions. In this paper we will determine how many of these combinations are solutions. Disregarding a brute force attempt to nd all solutions, we begin by describing this game by a system of polynomial equations.

2. DESCRIBING THE GAME

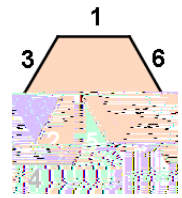


Figure 1: Nut B_0 in initial rotation state

2.1 Notation and Description
Each nut has a particular ordering of one through six. I will refer to the ordering of a speci c nut by a 6-tuple, headed by any number with subsequent numbers listed clockwise. The rst entry of the 6-tuple will correspond to number located on the north side of the nut, which we call position 0. The second entry will correspond to the number located on the east-north-east side of the nut, which we call position 1. Following entries will correspond to the next side moving clockwise up to position 5 corresponding to west-north-west.

If the zero entry of the 6-tuple is 1, then we shall call that the initial rotation state of the nut . For instance, in Figure 1, the second entry of B_0 is 2 and the fourth entry is 5. In a randomly assigned order, here are the de nitions for each nut in the initial rotation state: $B_0 = (1; 6; 2; 4; 5; 3)$, $B_1 = (1; 4; 6; 2; 3; 5)$, $B_2 = (1; 6; 5; 3; 2; 4)$, $B_3 = (1; 4; 3; 6; 5; 2)$, $B_4 = (1; 2; 3; 4; 5; 6)$, $B_5 = (1; 6; 4; 2; 5; 3)$, $B_6 = (1; 6; 5; 4; 3; 2)$.

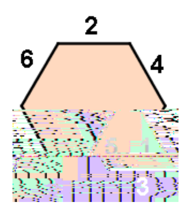


Figure 2: Nut B_0 rotated 4 times

example, B_0 rotated four times would look like Figure 2. The corresponding 6-tuple for B_0 rotated four times would be $(2; 4; 5; 3; 1; 6)$:

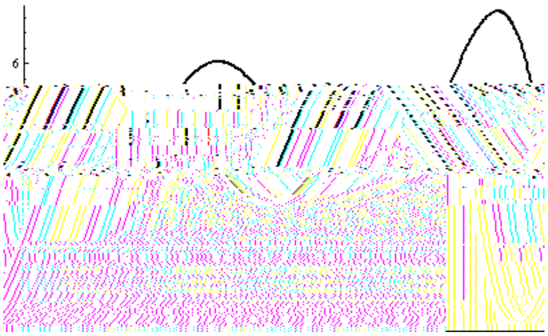


Figure 5: $f_{0;1}(x)$ on $[0;5]$

4.3 Finding a Solution

We have 27 equations of degree 16 or less in 16 variables with rational coe cients and we want to nd a solution to system. If we multiply an equation in the system by some $K \in R[g_0; g_1; g_2; g_3; g_4; g_5; g_6; p_2; p_3; p_4; p_5; a; b; c; d]$ or

Proof. Using a computer algebra system, the Gröbner basis of $I(V(I))$ is $< 1+12d; 1+24c; 1+120b; \ 1+720a; \ 4+p_6; \ 3+p_5; \ 3+p_4; p_3; \ 3+p_2; \ 6+g_6; \ 5+g_5; \ 4+g_4; \ 3+g_3; \ 2+g_2; \ 1+g_1; g_0 >:$ □

Our goal was to nd a Gröbner basis for I, but at best we have found a Gröbner basis that contains I and has the same variety.

Theorem 4.3. $< I > \ < 1+12d; 1+24c; 1+120b; \ 1+720a; \ 4+p_6; \ 3+p_5; \ 3+p_4; p_3; \ 3+p_2; \ 6+g_6; \ 5+g_5; \ 4+g_4; \ 3+g_3; \ 2+g_2; \ 1+g_1; g_0 >$

Proof. The proof follows directly from Lemma 7 on page 34 of [2]. □

We hope in the future to be able to show that the above is an equality and show that $< 1+12d; 1+24c; 1+120b; \ 1+720a; \ 4+p_6; \ 3+p_5; \ 3+p_4; p_3; \ 3+p_2; \ 6+g_6; \ 5+g_5; \ 4+g_4; \ 3+g_3; \ 2+g_2; \ 1+g_1; g_0 >$ is a Gröbner basis for our ideal.

5. SOLVING CIPRA'S PROBLEM

One interesting application to this technique of describing a combinatorics problem as a system of polynomial equations is Barry Cipra's Problem featured in [1]. There are sixteen distinct squares to be arranged on a four by four grid. Each square contains a distinct combination of a horizontal line through the center, a vertical line through the center, an up-right diagonal through the center, and a down-right diagonal through the center. Each of these squares is to be placed on the grid, rotations not allowed, such that all horizontal, diagonal, and vertical lines are unbroken.

Solution redundancy is di cult to avoid in this puzzle because some of the squares are 90 degree and 180 degree rotations of other squares. Since some squares have fournd a dp

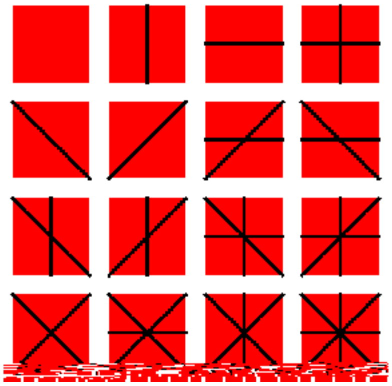


Figure 9: All of the squares in Cipra's Puzzle

these systems using ideals, varieties, and Grobner bases, but that is dependent on the power of computing available to calculate the Grobner bases. Lacking high-powered computing, we still are able to calculate a solution using Mathematica's Reduce function iteratively.

7. REFERENCES