AN APPLICATION OF THE IMPLICIT FUNCTION THEOREM TO COMPARATIVE STATICS ANALYSIS

ABSTRACT. Comparative statics analysis is concerned with the comparison of equilibriums that are associated with different sets of values of exogenous variables (parameters). [redacted]

policy implications of economic models are generated by comparative static analysis. [redacted]

Theorem. *[p. 63, 1] Let A be open in $\mathbb{R}^n$; let $f : A \to \mathbb{R}^n$ be of class C^r, meaning the first r derivatives of f exist and are continuous. If $Df(\overrightarrow{x})$ is non-singular at some point $\overrightarrow{a}$ in A, then there is a neighborhood U of the point $\overrightarrow{a}$ such that f carries U in a one-to-one fashion onto an open set V of $\mathbb{R}^n$ and the inverse function $f^{-1} : V \to U$ is of class C^r.*

[illegible]

at each point of A, it does not imply that f is (globally) one-to-one on all of A.

3. Implicit function

Definition. An equation of the form

$$f(x, y) = 0$$

[illegible]

Then there is a neighborhood B of $\overrightarrow{a}$ in $\mathbb{R}^k$ and a unique continuous function $g : B \to \mathbb{R}^n$ such that $g(\overrightarrow{a}) = \overrightarrow{b}$ and

$$f(\overrightarrow{x}, g(\overrightarrow{x})) = \overrightarrow{0}$$

Moreover,

$$Dg(\overrightarrow{x}) = -\left[\frac{\partial f}{\partial \overrightarrow{y}}(\overrightarrow{x}, g(\overrightarrow{x}))\right]^{-1} \cdot \frac{\partial f}{\partial \overrightarrow{x}}(\overrightarrow{x}, g(\overrightarrow{x})).$$

Proof. Define $F : A \to \mathbb{R}^{k+n}$ by the equation

$$F(\overrightarrow{x}, \overrightarrow{y}) = (\overrightarrow{x}, f(\overrightarrow{x}, \overrightarrow{y})).$$

$$
\begin{aligned}
detDF(\overrightarrow{x}, \overrightarrow{y}) &= detJ \\
&= detJ_{11} \\
&= detJ_{22} \\
&= \ldots \\
&= detJ_{kk}
\end{aligned}
$$

So that the determinant of Jacobian matrix of function F is non-singular at the point $(\overrightarrow{a}, \overrightarrow{b})$

Thus, there exists a continuous function $g : B \to \mathbb{R}^n$ such that $g(\overrightarrow{a}) = \overrightarrow{b}$ and

$$f(\overrightarrow{x}, g(\overrightarrow{x})) = \overrightarrow{0} \text{ for all } \overrightarrow{x} \in \text{B}.$$

[illegible]

$(x_1, ..., x_k) \in \mathbb{R}^k$ is

$$
\begin{aligned}
Dp(\overrightarrow{x}) &= \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_k} \\ \vdots & (I_k) & \\ \frac{\partial x_k}{\partial x_1} & \cdots & \frac{\partial x_k}{\partial x_k} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_k} \\ \vdots & (Dg(\overrightarrow{x})) & \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_k} \end{bmatrix} \\
&= \begin{bmatrix} I_k \\ Dg(\overrightarrow{x}) \end{bmatrix}.
\end{aligned}
$$

Substitute $Df(p(\overrightarrow{x})$ and $Dp(\overrightarrow{x})$ to the equation 4.3, we get

$$
\begin{aligned}
0 &= \begin{bmatrix} \frac{\partial f}{\partial \overrightarrow{x}}(p(\overrightarrow{x})) & \frac{\partial f}{\partial \overrightarrow{y}}(p(\overrightarrow{x})) \end{bmatrix} . \begin{bmatrix} I_k \\ Dg(\overrightarrow{x}) \end{bmatrix} \\
&= \frac{\partial f}{\partial \overrightarrow{x}}(p(\overrightarrow{x})) + \frac{\partial f}{\partial \overrightarrow{y}}(p(\overrightarrow{x})).Dg(\overrightarrow{x}).
\end{aligned}
$$

So,

$$
Dg(\overrightarrow{x}) = -\left[\frac{\partial f}{\partial \overrightarrow{y}}(\overrightarrow{x}, g(\overrightarrow{x}))\right]^{-1} . \frac{\partial f}{\partial \overrightarrow{x}}(\overrightarrow{x}, g(\overrightarrow{x})).
$$

□

The Implicit Function Theorem is extremely useful in economics to have a quick analysis when the real money supply M is equal to the real demand for money, which depends on national income y and the real interest rate r. We assume that the real demand for money would increase if national income increased and the real interest rate decreased (i.e. $\frac{\partial L}{\partial y} > 0$ and $\frac{\partial L}{\partial r} < 0$).

A short summary of our model assumptions is followed

$$
\begin{aligned}
1 > \frac{\partial C}{\partial y} &> 0 \\
\frac{\partial C}{\partial r} &< 0 \\
\frac{\partial I}{\partial r} &> 0 \\
\frac{\partial L}{\partial y} &> 0 \\
\frac{\partial L}{\partial r} &< 0.
\end{aligned}
\tag{5.1}
$$

Analysis of this model consists of examining the impact of changes in the exogenous variables G, M on the dependent variables y, r. Designating two functions of equilibrium, we

So,

$$Df(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial u} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1}{\partial G} & \frac{\partial f_1}{\partial M} \end{bmatrix}$$

Then, the function $f = (f_1, f_2)$ from $\mathbb{R}^4$ to $\mathbb{R}^2$ describing equilibrium in the goods and money markets is written as

$$\begin{aligned} f_1(G, M, r, y) &= y - C(y, r) - I(r) - G \\ &= y - (a_C y + b_C r + c_C) - (b_I r + c_I) - G \\ &= (1 - a_C)y - (b_C + b_I)r - (c_C + c_I) - G \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} f_2(G, M, r, y) &= L(y, r) - M \\ &= a_L y + b_L r + c_L - M \\ &= 0. \end{aligned}$$

So,

$$G = (1 - a_C)y - (b_C + b_I)r - (c_C + c_I) \tag{6.1}$$

$$M = a_L y + b_L r + c_L. \tag{6.2}$$

To derive r as a function of G and M, we multiply both side of the equation 6.1 with a_L and the equation 6.2 with $(1 - a_C)$

$$\begin{aligned} Ga_L &= (1 - a_C)a_L y - (b_C + b_I)a_L r - (c_C + c_I)a_L \\ M(1 - a_C) &= (1 - a_C)a_L y + (1 - a_C)b_L r + c_L(1 - a_C) \end{aligned}$$

then,

$$\begin{aligned} Ga_L - M(1 - a_C) &= (1 - a_C)a_L y - (b_C + b_I)a_L r - (c_C + c_I)a_L \\ &\quad - (1 - a_C)a_L y - (1 - a_C)b_L r - c_L(1 - a_C) \\ &= -r[(b_C + b_I)a_L + (1 - a_C)b_L] - [(c_C + c_I)a_L + c_L(1 - a_C)]. \end{aligned}$$

So, we can directly derive r as a function of G and M

$$r = \frac{-a_L}{[\text{illegible}]} G + \frac{1 - a_C}{[\text{illegible}]} M [\text{illegible}] \frac{(c_C + c_I)a_L + c_L(1 - a_C)}{[\text{illegible}]}$$

So, we can explicitly solve for y as a function of G and M

b_L [illegible] $(b_C + b_I)$ [illegible] $(c_C + c_I)b_L - c_L(b_C + b_I)$

[illegible]

$$\begin{bmatrix} \frac{\partial r}{\partial G} & \frac{\partial r}{\partial M} \end{bmatrix} \quad 1 \quad \begin{bmatrix} -a_L & (1-a_C) \end{bmatrix}$$

means

$$\frac{\partial r}{\quad} = \frac{-a_L}{\quad} > 0$$

$$\frac{\partial r}{\partial M} = \frac{(1-a_C)}{a_L(b_C+b_I)+(1-a_C)b_L} < 0$$

$$\frac{\partial y}{\partial G} = \frac{b_L}{a_L(b_C+b_I)+(1-a_C)b_L} > 0$$

$$\frac{\partial y}{\partial M} = \frac{(b_C+b_I)}{a_L(b_C+b_I)+(1-a_C)b_L} > 0.$$

Indeed, the result derived by the Implicit Function Theorem coincides with the "explicit"

Since we assume $\frac{\partial d}{\partial p} < 0$ and $\frac{\partial s}{\partial p} > 0$, we get

$$\frac{\partial f}{\partial p} = \frac{\partial d}{\partial p} - \frac{\partial s}{\partial p} < 0.$$

Moreover, when the price of new cars is at higher level, a fixed amount of an increase in new cars price would make smaller impact on the demand for used cars. For example, a price of new cars would increase when the price of used car increases, which coincides with [redacted]

[redacted]

[redacted]

Theorem. *If the demand and supply functions are linear, the relative price ratio is constant on the equilibrium path.*

Proof. Suppose the demand function for used car is

$$D_u(P_n, P_u) = aP_n - bP_u$$

where a, b are positive. The supply function is

$$S_u(P_n, P_u) = -cP_n + dP_u$$

where c, d are positive. [redacted]

[redacted]

Proof. Suppose the demand function has the form

$$D_u(P_n, P_u) = h(P_n)k(P_u)$$

$$S_u(P_n, P_u) = p(P_n)q(P_u)$$

Since demand and supply function are always positive, without loss of generality, suppose h, k, p, q are positive functions. Then, by the above lemma, since h, k, p, q are homogenous functions of one variable, we can rewrite our demand and supply functions in a specific form, such as

$$\begin{aligned} D_u(P_n, P_u) &= cP_n^a P_u^b \\ S_u(P_n, P_u) &= dP_n^e P_u^k \end{aligned}$$

where c and d are positive.

Moreover, from the assumption (8.1)

$$\begin{aligned} \frac{\partial D_u}{\partial P_n} &= caP_n^{a-1}P_u^b > 0 \\ \frac{\partial D_u}{\partial P_u} &= cbP_n^a P_u^{b-1} < 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial S_u}{\partial P_n} &= deP_n^{e-1}P_u^k < 0 \\ \frac{\partial S_u}{\partial P_u} &= dkP_n^e P_u^{k-1} > 0 \end{aligned}$$

we know $a, k > 0$ and $b, e < 0$.

Let $\alpha(P_n, P_u) = \dfrac{P_n}{P_u}$ be the function of the relative price ratio. If $\alpha(P_n, P_u)$ is a constant, then substitute $P_n = \alpha P_u$ in our market equilibrium condition, $D_u = S_u$, we get

$$c\alpha^a P_u^{a+b} = d\alpha^e P_u^{e+k}$$

Proof. Recall our model

$$
\begin{aligned}
D_u(P_n, P_u) &= cP_n^a P_u^b \\
S_u(P_n, P_u) &= dP_n^e P_u^k
\end{aligned}
$$

where $a, k, c, d > 0$, and $b, e < 0$.

Then the condition for the market equilibrium $D_u = S_u$ means $cP_n^a P_u^b = dP_n^e P_u^k$, or

$$P_n^{a-e} = \frac{d}{c} P_u^{k-b}.$$

$d(\partial D_u/\partial P_u)$ $d(\partial D_u/\partial P_n)$

[redacted] $d(\partial D_u/\partial P_u)$ [redacted]

second part, $\frac{dP_u}{d(P_n/P_u)}$, will be examined below.

[redacted]

Recall that $\frac{\partial D_u}{\partial P_n} = caP_n^{a-1}P_u^b$, then we substitute $P_n = (\frac{d}{c}P_u^{k-b})^{\frac{1}{a-e}}$ in equation 8.4 into the function and get

$$\begin{aligned}
\frac{\partial D_u}{\partial P_n} &= caP_n^{a-1}P_u^b \\
&= ca(\frac{d}{c}P_u^{k-b})^{\frac{a-1}{a-e}}P_u^b \\
&= ca(\frac{d}{c})^{\frac{a-1}{a-e}}P_u^{\frac{(k-b)(a-1)+b(a-e)}{a-e}} \\
&= ca(\frac{d}{c})^{\frac{a-1}{a-e}}P_u^{\frac{ka-ba-k+b+ba-be}{a-e}}
\end{aligned}$$

[illegible]

The derivation of ∂D_u with [illegible]

[illegible]

$$\frac{d\ (\partial D_u/\partial P_n)}{dP_u} = ca(\frac{d}{c})^{\frac{a-1}{a-e}}\frac{ka-k+b-be}{a-e}P_u^{\frac{ka-k+b-be}{a-e}-1}.$$

[illegible]

full proof of the Implicit Function Theorem is presented in section 4. Basically, the Theorem consists of two main parts: conditions (continuity and non-singularity) for the existence and uniqueness of the implicit function, and the formula for its derivative. The proof technique

requires a broad background in mathematics, as we need to use multivariable calculus, lin- [illegible]